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著者	Cheng Qing-Ming, Choi Soon Meen
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COMPLETE SPACE-LIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF AN INDEFINITE SPACE FORM

By

Qing-ming CHENG* and Soon Meen CHOI

1. Introduction.

Let $M_p^{n+p}(c)$ be an $(n+p)$ -dimensional connected indefinite Riemannian manifold of index p and of constant curvature c , which is called an *indefinite space form of index p* . According to $c>0$, $c=0$ or $c<0$ it is denoted by $S_p^{n+p}(c)$, R_p^{n+p} or $H_p^{n+p}(c)$. A submanifold M of an indefinite space form $M_p^{n+p}(c)$ is said to be *space-like* if the induced metric on M from that of the ambient space is positive definite. It is pointed out by some physicians that space-like hypersurfaces with constant mean curvature of arbitrary spacetimes get interested in relativity theory and an entire space-like hypersurface with constant mean curvature of an indefinite space form are studied by many authors (for examples: [1], [2], [3], [4], [7], [12] and so on).

Now, for a complete space-like submanifold M with parallel mean curvature vector of $S_p^{n+p}(c)$, it is also seen by the first author [5] that M is totally umbilic if $n=2$ and $h^2 \leq 4c$ or if $n>2$ and $h^2 < 4(n-1)c$, where H denotes the mean curvature, i.e., the norm of the mean curvature vector and $h=nH$. On the other hand, the first author and Nakagawa [6] investigated the total umbilicness of such hypersurfaces from the different point of view. They proved that the squared norm S of the second fundamental form of M is bounded from above by $S_+(1)$ and if $\sup S < S_-(1)$ and $H^2 \leq c$, then M is totally umbilic, where

$$S_{\pm}(p) = -pnc + \frac{nh^2 \pm (n-2)\sqrt{h^4 - 4(n-1)ch^2}}{2(n-1)}.$$

In this paper, we research the similar problem to the above property for the complete space-like submanifolds with parallel mean curvature vector of an indefinite space form. That is, we prove the following

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THEOREM 1. Let M be an n -dimensional complete space-like submanifold with parallel mean curvature vector of an indefinite space form $M_p^{n+p}(c)$. If the one of the following conditions is satisfied:

- (1) $c \leq 0$,
- (2) $c > 0$ and $n^2 H^2 \geq 4(n-1)c$,

then

$$(1.1) \quad S \leq S_+(p) + K(p),$$

where $K(p)$ is a constant defined by

$$K(p) = (p-1)H \{nH + \sqrt{n(n-1)\{S_+(1) - nH^2\}}\}.$$

THEOREM 2. The hyperbolic cylinder $H^1(c_1) \times \mathbf{R}^{n-1}$ in \mathbf{R}_1^{n+1} is the only complete connected space-like n -dimensional submanifolds with parallel mean curvature vector of \mathbf{R}_p^{n+p} satisfying $S = S_+(p) + K(p)$.

THEOREM 3. The hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$ of $H_1^{n+1}(c)$ and the maximal submanifolds $H^{n_1}(c_1) \times \cdots \times H^{n_{p+1}}(c_{p+1})$ of $H_p^{n+p}(c)$ are the only complete connected space-like n -dimensional submanifolds with parallel mean curvature vector satisfying $S = S_+(p) + K(p)$, where $c_r = (n/n_r)c$ and $\sum_{r=1}^{p+1} n_r = n$ in the latter case.

2. Standard models.

This section is concerned with some standard models of complete space-like submanifolds with parallel mean curvature vector of an indefinite space form $M_p^{n+p}(c)$, $c \leq 0$. In particular, we only consider non-totally umbilic cases. Moreover, the squared norms of the second fundamental forms of such standard models are calculated. Without loss of generality, an $(n+p)$ -dimensional indefinite Euclidean space \mathbf{R}_p^{n+p} of index $p(\geq 1)$ can be first regarded as a product manifold of

$$\mathbf{R}_1^{n_1+1} \times \cdots \times \mathbf{R}_1^{n_{p+1}+1} \times \mathbf{R}^m,$$

where $\sum_{r=1}^p n_r + m = n$. With respect to the standard orthonormal basis of \mathbf{R}_p^{n+p} a class of space-like submanifolds

$$H^{n_1}(c_1) \times \cdots \times H^{n_p}(c_p) \times \mathbf{R}^m$$

of \mathbf{R}_p^{n+p} is defined as the Pythagorean product

$$H^{n_1}(c_1) \times \cdots \times H^{n_p}(c_p) \times \mathbf{R}^m \\ = \left\{ (x_1, \dots, x_{p+1}) \in \mathbf{R}_p^{n+p} = \mathbf{R}_1^{n_1+1} \times \cdots \times \mathbf{R}_1^{n_{p+1}+1} \times \mathbf{R}^m : |x_r|^2 = -\frac{1}{c_r} > 0 \right\},$$

where $r=1, \dots, p$ and $||$ denotes the norm defined by the product on the Minkowski space R_1^{p+1} which is given by $\langle x, x \rangle = -(x_0)^2 + \sum_{j=1}^k (x_j)^2$. The mean curvature vector h of M is given by

$$(2.1) \quad h = -\frac{1}{n} \sum_{r=1}^p n_r c_r x_r$$

at $(x_1, \dots, x_{p+1}) \in M$, which is parallel in the normal bundle of M . The number $S_+(1)$ and the squared norm S of the second fundamental form are given by

$$(2.2) \quad S_+(1) = n^2 H^2 = -\sum_{r=1}^p n_r^2 c_r, \quad S = -\sum_{r=1}^p n_r c_r.$$

Then we get

$$S_+(p) + K(p) = p n^2 H^2 = -p \sum_{r=1}^p n_r^2 c_r \geq S,$$

where the equality holds if and only if $p=1$ and $n_1=1$.

Next we consider an n -dimensional space-like submanifold of $H_p^{n+p}(c)$, $p \geq 1$. Without loss of generality, an $(n+p+1)$ -dimensional indefinite Euclidean space R_{p+1}^{n+p+1} of index $(p+1)$ can be first regarded as a product manifold of

$$R_1^{n_1+1} \times \dots \times R_1^{n_{p+1}+1},$$

where $\sum_{r=1}^{p+1} n_r = n$. With respect to the standard orthonormal basis of R_{p+1}^{n+p+1} a class of space-like submanifolds

$$H^{n_1}(c_1) \times \dots \times H^{n_{p+1}}(c_{p+1})$$

of R_{p+1}^{n+p+1} is defined as the Pythagorean product

$$\begin{aligned} & H^{n_1}(c_1) \times \dots \times H^{n_{p+1}}(c_{p+1}) \\ &= \left\{ (x_1, \dots, x_{p+1}) \in R_{p+1}^{n+p+1} = R_1^{n_1+1} \times \dots \times R_1^{n_{p+1}+1} : |x_r|^2 = -\frac{1}{c_r} > 0 \right\}, \end{aligned}$$

where $r=1, \dots, p+1$. The mean curvature vector h of M is given by

$$(2.3) \quad h = -\frac{1}{n} \sum_{r=1}^{p+1} (n_r c_r x_r) + c x$$

at $x = (x_1, \dots, x_{p+1}) \in M$, which is parallel in the normal bundle of M . The norm H of the mean curvature vector h and the squared norm S of the second fundamental form are given by

$$(2.4) \quad h^2 = n^2 H^2 = n^2 c - \sum_{r=1}^{p+1} n_r^2 c_r, \quad S = \sum_{r=1}^{p+1} n_r (c - c_r) = n c - \sum_{r=1}^{p+1} n_r c_r.$$

When M is maximal, it satisfies $n_r c_r = n c$ for any index r by (2.3), which yields $S = -p n c$. Then we get $S_+(p) + K(p) - S = 0$, because of $S_+(p) = -p n c$ and $K(p) = 0$.

Suppose that $H \neq 0$. By a theorem of Ki, Kim and Nakagawa [9], if $p=1$, then we have $S_+(1)-S=0$. On the other hand, we have $S_+(1)>h^2-nc$, because of $c<0$. So it is seen that if $p \geq 2$, then we obtain

$$S_+(p)+K(p)-S>h^2-pnc+(p-1)h^2-S=ph^2-pnc-S \geq 0$$

by (2.4). In order to prove the last inequality, the following lemma is prepared. The proof of this lemma is the only calculus and hence it is omitted.

LEMMA 2.1. *Let a_1, \dots, a_{p+1} be numbers not less than 1 satisfying $\sum a_r = n$ and b_1, \dots, b_{p+1} be negative numbers satisfying $\sum (1/b_r) = (1/b)$. Then we have*

$$\sum \{a_r - p(a_r)^2\} b_r \geq n(p+1 - pn)b.$$

3. Preliminaries.

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category. Let $M_p^{n+p}(c)$ be an $(n+p)$ -dimensional indefinite Riemannian manifold of constant curvature c whose index is p , which is called *an indefinite space form of constant curvature c and with index p* . Let M be an n -dimensional submanifold of an $(n+p)$ -dimensional indefinite space form $M_p^{n+p}(c)$ of index p . The submanifold M is said to be *space-like* if the induced metric on M from that of the ambient space is positive definite. We choose a local field of orthonormal frames e_1, \dots, e_{n+p} adapted to the indefinite Riemannian metric of $M_p^{n+p}(c)$ and the dual coframes $\omega_1, \dots, \omega_{n+p}$ in such a way that, restricted to the submanifold M , e_1, \dots, e_n are tangent to M . Then connection forms $\{\omega_{AB}\}$ of $M_p^{n+p}(c)$ are characterized by the structure equations

$$(3.1) \quad \begin{cases} d\omega_A + \sum \varepsilon_B \omega_{AB} \wedge \omega_B = 0, & \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} + \sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\ \Omega_{AB} = -\frac{1}{2} \sum \varepsilon_C \varepsilon_D R'_{ABCD} \omega_C \wedge \omega_D, \end{cases}$$

$$(3.2) \quad R'_{ABCD} = c \varepsilon_A \varepsilon_B (\delta_{AD} \delta_{BC} - \delta_{AC} \delta_{BD}),$$

where $\varepsilon_A = 1$ for an index $A \leq n$, $\varepsilon_A = -1$ for an index $A \geq n+1$, and Ω_{AB} (resp. R'_{ABCD}) denotes the indefinite Riemannian curvature form (resp. the components of the indefinite Riemannian curvature tensor R') of $M_p^{n+p}(c)$. Therefore the components of the Ricci curvature tensor Ric' and the scalar curvature r' of $M_p^{n+p}(c)$ are given as

$$R'_{AB} = c(n+p-1)\varepsilon_A \delta_{AB}, \quad r' = (n+p)(n+p-1)c.$$

In the sequel, the following convention on the range of indices is used, unless otherwise stated :

$$1 \leq A, B, \dots \leq n+p; \quad 1 \leq i, j, \dots \leq n; \quad n+1 \leq \alpha, \beta, \dots \leq n+p.$$

We agree that the repeated indices under a summation sign without indication are summed over the respective range. The canonical forms $\{\omega_A\}$ and the connection forms $\{\omega_{AB}\}$ restricted to M are also denoted by the same symbols. We then have

$$(3.3) \quad \omega_\alpha = 0 \quad \text{for } \alpha = n+1, \dots, n+p.$$

We see that e_1, \dots, e_n is a local field of orthonormal frames adapted to the induced Riemannian metric on M and $\omega_1, \dots, \omega_n$ is a local field of its dual coframes on M . It follows from (3.1), (3.3) and Cartan's lemma that we have

$$(3.4) \quad \omega_{\alpha i} = \sum h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

The second fundamental form α and the mean curvature vector h of M are defined by

$$\alpha = -\sum h_{ij}^\alpha \omega_i \omega_j e_\alpha, \quad h = -\frac{1}{n} \sum (\sum_i h_{ii}^\alpha) e_\alpha.$$

The mean curvature H is defined by

$$(3.5) \quad H = |h| = \frac{1}{n} \sqrt{\sum (\sum_i h_{ii}^\alpha)^2}.$$

Let $S = \sum (h_{ij}^\alpha)^2$ denote the squared norm of the second fundamental form α of M . The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations

$$(3.6) \quad \begin{cases} d\omega_i + \sum \omega_{ij} \wedge \omega_j = 0, & \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ \Omega_{ij} = -\frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \end{cases}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) of M . Therefore, from (3.1) and (3.6), the Gauss equation is given by

$$(3.7) \quad R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) - \sum (h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha).$$

The components of the Ricci curvature Ric and the scalar curvature r are given by

$$(3.8) \quad R_{jk} = (n-1)c\delta_{jk} - \sum h_{ji}^\alpha h_{jk}^\alpha + \sum h_{ji}^\alpha h_{ik}^\alpha,$$

$$(3.9) \quad r = n(n-1)c - n^2 H^2 + \sum (h_{ij}^\alpha)^2.$$

We also have

$$(3.10) \quad d\omega_{\alpha\beta} - \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} = -\frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j,$$

where

$$R_{\alpha\beta ij} = -\sum (h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta).$$

The Codazzi equation and the Ricci formula for the second fundamental form are given by

$$(3.11) \quad h_{ijk}^\alpha - h_{ikj}^\alpha = 0,$$

$$(3.12) \quad h_{ijk}^\alpha - h_{ijl}^\alpha = -\sum h_{im}^\alpha R_{mjkl} - \sum h_{mj}^\alpha R_{mikl} + \sum h_{ij}^\beta R_{\beta\alpha kl},$$

where h_{ijk}^α and h_{ijl}^α denote the components of the covariant differentials $\nabla\alpha$ and $\nabla^2\alpha$ of the second fundamental form, respectively. The Laplacian Δh_{ij}^α of the components h_{ij}^α of the second fundamental form α is given by

$$\Delta h_{ij}^\alpha = \sum h_{ijk}^\alpha.$$

From (3.12) we get

$$(3.13) \quad \Delta h_{ij}^\alpha = \sum_k h_{kki}^\alpha - \sum h_{km}^\alpha R_{mijk} - \sum h_{mi}^\alpha R_{mkjk} + \sum h_{ki}^\beta R_{\beta\alpha jk}.$$

The following generalized maximum principle due to Omori [11] and Yau [15] will play an important role in this paper.

THEOREM 3.1. *Let M be an n -dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2 -function bounded from above on M , then for any $\varepsilon > 0$, there exists a point p in M such that*

$$F(p) + \varepsilon > \sup F, \quad |\text{grad } F|(p) < \varepsilon, \quad \Delta F(p) < \varepsilon.$$

The following lemma is already known.

LEMMA 3.2. *Let a_1, \dots, a_n be real numbers satisfying $\sum a_i = 0$ and $\sum a_i^2 = k^2$ for $k > 0$. Then we have*

$$|\sum a_i^3| \leq (n-2) \sqrt{\frac{1}{n(n-1)}} k^3,$$

where the equality holds if and only if $n-1$ of them are equal with each other.

4. Pseudo-umbilic submanifolds.

Let M be an n -dimensional space-like submanifold with parallel mean curvature vector \mathbf{h} of an indefinite space form $M_p^{n+p}(c)$. Because the mean curvature vector is parallel, the mean curvature is constant. Suppose that

$H \neq 0$. We choose e_{n+1} in such a way that its direction coincides with that of the mean curvature vector. Then it is easily seen that we have

$$(4.1) \quad \omega_{\alpha n+1} = 0, \quad H = \text{constant},$$

$$(4.2) \quad H^\alpha H^{n+1} = H^{n+1} H^\alpha,$$

$$(4.3) \quad \text{tr} H^{n+1} = nH, \quad \text{tr} H^\alpha = 0$$

for any $\alpha \neq n+1$, where H^α denotes an $n \times n$ symmetric matrix (h_{ij}^α) .

A submanifold M is said to be *pseudo-umbilic*, if it is umbilic with respect to the direction of the mean curvature vector \mathbf{h} , that is,

$$(4.4) \quad h_{ij}^{n+1} = H \delta_{ij}.$$

We denote by μ an $n \times n$ symmetric matrix with $\mu_{ij} = h_{ij}^{n+1} - H \delta_{ij}$. Then we have

$$(4.5) \quad \text{tr} \mu = 0, \quad |\mu|^2 = \text{tr}(\mu^2) = \sum (\mu_{ij})^2 = \text{tr}(H^{n+1})^2 - nH^2.$$

So the pseudo-umbilic submanifolds are characterized by the property $\mu = 0$. A non-negative function τ is defined by $\tau^2 = \sum_{\beta \neq n+1} (h_{ij}^\beta)^2$. We then have

$$(4.6) \quad S = |\mu|^2 + \tau^2 + nH^2.$$

Hence it is seen that $|\mu|^2$ as well as τ^2 are independent of the choice of the frame fields and they are functions defined globally on M .

PROPOSITION 4.1. *Let M be n -dimensional complete space-like submanifold with parallel mean curvature vector of an indefinite space form $S_p^{n+p}(c)$. If it satisfies*

$$n^2 c \geq n^2 H^2 \geq 4(n-1)c, \quad S \leq S_-(1),$$

then M is pseudo-umbilic, where H denotes the mean curvature, i.e., the norm of the mean curvature vector.

PROOF. In order to prove this property it suffices to show $\mu = 0$. From (3.13), the Gauss equation (3.7) and (3.10), we have

$$(4.7) \quad \begin{aligned} \Delta h_{ij}^{n+1} = & nch_{ij}^{n+1} - ncH\delta_{ij} + \sum h_{km}^{n+1} h_{mk}^\beta h_{ij}^\beta - 2 \sum h_{ik}^\beta h_{km}^{n+1} h_{mj}^\beta \\ & + \sum h_{im}^{n+1} h_{mk}^\beta h_{kj}^\beta - nH \sum h_{im}^{n+1} h_{mj}^{n+1} + \sum h_{ik}^\beta h_{km}^\beta h_{mj}^{n+1}. \end{aligned}$$

Accordingly we obtain from (4.2)

$$\begin{aligned} \frac{1}{2} \Delta |\mu|^2 = & \sum (h_{ij}^{n+1})^2 + nc \sum (h_{ij}^{n+1})^2 - n^2 c H^2 \\ & + \sum h_{km}^{n+1} h_{mk}^\beta h_{ij}^\beta h_{ij}^{n+1} - 2 \sum h_{ik}^\beta h_{km}^{n+1} h_{mj}^\beta h_{ij}^{n+1} + \sum h_{im}^{n+1} h_{mk}^\beta h_{kj}^\beta h_{ij}^{n+1} \end{aligned}$$

$$-nH \sum h_{im}^{n+1} h_{mj}^{n+1} h_{ij}^{n+1} + \sum h_{ik}^{\beta} h_{km}^{\beta} h_{mj}^{n+1} h_{ij}^{n+1}$$

and hence we see

$$(4.8) \quad \begin{aligned} \frac{1}{2} \Delta |\mu|^2 &= \sum (h_{ij}^{n+1})^2 + nc \sum (h_{ij}^{n+1})^2 \\ &\quad - n^2 c H^2 - nH \operatorname{tr}(H^{n+1})^3 - \sum_{\beta \neq n+1} \operatorname{tr}(H^{n+1} H^{\beta} - H^{\beta} H^{n+1})^2 \\ &\quad + \{\operatorname{tr}(H^{n+1})^2\}^2 + \sum_{\beta \neq n+1} \{\operatorname{tr}(H^{n+1} H^{\beta})\}^2. \end{aligned}$$

On the other hand, because of

$$\operatorname{tr}(H^{n+1})^3 = \operatorname{tr} \mu^3 + 3H \{\operatorname{tr}(H^{n+1})^2 - nH^2\} + nH^3,$$

we get

$$(4.9) \quad \begin{aligned} \frac{1}{2} \Delta |\mu|^2 &\geq (|\mu|^2 + nH^2)^2 - nH \{\operatorname{tr} \mu^3 + 3H |\mu|^2 + nH^3\} + nc |\mu|^2 \\ &= |\mu|^2 (|\mu|^2 + nc - nH^2) - nH \operatorname{tr} \mu^3. \end{aligned}$$

Because of $\operatorname{tr} \mu = 0$, we can apply Lemma 3.2 to the eigenvalues of μ and obtain

$$(4.10) \quad |\operatorname{tr} \mu^3| \leq \frac{n-2}{\sqrt{n(n-1)}} |\mu|^3.$$

Hence we obtain

$$(4.11) \quad \frac{1}{2} \Delta |\mu|^2 \geq |\mu|^2 \left(|\mu|^2 - nH \frac{n-2}{\sqrt{n(n-1)}} |\mu| + nc - nH^2 \right),$$

where we have used (4.9) and (4.10). From (3.8) we know that the Ricci curvature of M is bounded from below. Putting $F = -1/\sqrt{|\mu|^2 + a}$ for any positive number a . Since M is complete and space-like, we can apply the Generalized Maximum Principle (Theorem 3.1) to the function F . For any given positive number $\varepsilon > 0$, there exists a point p at which F satisfies

$$(4.12) \quad \sup F < F(p) + \varepsilon, \quad |\operatorname{grad} F|(p) < \varepsilon, \quad \Delta F(p) < \varepsilon.$$

Consequently the following relationship

$$(4.13) \quad \frac{1}{2} F(p)^4 \Delta |\mu|^2(p) < 3\varepsilon^2 - F(p)\varepsilon$$

can be derived by the simple and direct calculations. For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \rightarrow 0$ ($m \rightarrow \infty$) and $\varepsilon_m > 0$, there exists a point sequence $\{p_m\}$ such that $\{F(p_m)\}$ converges to $F_0 = \sup F$ by (4.12). On the other hand, it follows from (4.13) that we have

$$(4.14) \quad \frac{1}{2} F(p_m)^4 \Delta |\mu|^2(p_m) < 3\varepsilon_m^2 - F(p_m)\varepsilon_m.$$

The right hand side of (4.14) converges to 0 because F is bounded. Accordingly,

for any positive number $\varepsilon > 0$ ($\varepsilon < 2$) there exists a sufficiently large integer m for which we have

$$F(p_m)^4 \Delta |\mu|^2(p_m) < \varepsilon.$$

Hence we get

$$(2-\varepsilon)|\mu|^4(p_m) - 2nH \frac{n-2}{\sqrt{n(n-1)}} |\mu|^3(p_m) \\ + 2(nc - nH^2 - \varepsilon a) |\mu|^2(p_m) - \varepsilon a^2 < 0.$$

Thus the sequence $\{|\mu|^2(p_m)\}$ is bounded and the definition of F gives rise to

$$(4.15) \quad \lim_{m \rightarrow \infty} |\mu|^2(p_m) = \sup |\mu|^2.$$

Therefore the supremum of F satisfies $F_0 = \sup F < 0$. According to (4.14) we have

$$(4.16) \quad \limsup_{m \rightarrow \infty} \Delta |\mu|^2(p_m) \leq 0.$$

Thus (4.11) and (4.16) yield

$$(4.17) \quad 0 \geq \sup |\mu|^2 \left(\sup |\mu|^2 - nH \frac{n-2}{\sqrt{n(n-1)}} \sup |\mu| + nc - nH^2 \right).$$

Taking account of (4.5) we have

$$(4.18) \quad \sup \Sigma (h_{ij}^{n+1})^2 = nH^2 \quad \text{or} \quad S_-(1) \leq \sup \Sigma (h_{ij}^{n+1})^2 \leq S_+(1),$$

from which combining with the assumption of Proposition 4.1 it follows that we have

$$\sup \Sigma (h_{ij}^{n+1})^2 = nH^2.$$

This means that $\mu = 0$ because of (4.5) and therefore M is pseudo-umbilic. \square

The inequality (4.17) holds on the space-like submanifold M of $M_p^{n+p}(c)$. Accordingly, in this case we have

$$(4.19) \quad \sup \Sigma (h_{ij}^{n+1})^2 = nH^2 \quad \text{or} \quad \sup \Sigma (h_{ij}^{n+1})^2 \leq S_+(1).$$

REMARK. When $p=1$, the hypersurface M becomes totally umbilic under the assumption of Proposition 4.1, which means that this property is a generalization of the theorem due to the first author and Nakagawa [6].

5. Proof of Theorem 1.

In this section the squared norm S of the second fundamental form of M is estimated from above. Let M be an n -dimensional space-like submanifold with parallel mean curvature vector h of an indefinite space form $M_p^{n+p}(c)$.

PROOF OF THEOREM 1. Because the mean curvature vector is parallel, the mean curvature is constant. If $H=0$, then from Theorem 1.1 due to Ishihara [8], we know that M is totally geodesic if $c \geq 0$ and $S \leq -npc$ if $c < 0$. Hence Theorem 1 is true. Next we may suppose $H \neq 0$. We choose e_{n+1} in such a way that its direction coincides with that of the mean curvature vector. Then we get (4.1), (4.2) and (4.3). From (3.13), the Gauss equation (3.7) and (3.10) we get

$$\begin{aligned} \frac{1}{2} \Delta \tau^2 &= \sum_{\alpha \neq n+1} (h_{ijk}^\alpha)^2 + \sum_{\alpha \neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &= \sum_{\alpha \neq n+1} (h_{ijk}^\alpha)^2 + nc\tau^2 + \sum_{\alpha \neq n+1} h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta h_{ij}^\alpha \\ &\quad - 2 \sum_{\alpha \neq n+1} h_{ik}^\beta h_{km}^\alpha h_{mj}^\beta h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{im}^\alpha h_{mk}^\beta h_{kj}^\beta h_{ij}^\alpha \\ &\quad - nH \sum_{\alpha \neq n+1} h_{im}^\alpha h_{mj}^{n+1} h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{ik}^\beta h_{km}^\beta h_{mj}^\alpha h_{ij}^\alpha, \end{aligned}$$

and hence we get

$$\begin{aligned} \frac{1}{2} \Delta \tau^2 &= \sum_{\alpha \neq n+1} (h_{ijk}^\alpha)^2 + nc\tau^2 + \sum_{\alpha, \beta \neq n+1} h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta h_{ij}^\alpha \\ &\quad - 2 \sum_{\alpha, \beta \neq n+1} h_{ik}^\beta h_{km}^\alpha h_{mj}^\beta h_{ij}^\alpha + \sum_{\alpha, \beta \neq n+1} h_{im}^\alpha h_{mk}^\beta h_{kj}^\beta h_{ij}^\alpha \\ (5.1) \quad &\quad + \sum_{\alpha, \beta \neq n+1} h_{ik}^\beta h_{km}^\beta h_{mj}^\alpha h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{km}^\alpha h_{mk}^{n+1} h_{ij}^{n+1} h_{ij}^\alpha \\ &\quad - 2 \sum_{\alpha \neq n+1} h_{ik}^{n+1} h_{km}^\alpha h_{mj}^{n+1} h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{im}^\alpha h_{mk}^{n+1} h_{kj}^{n+1} h_{ij}^\alpha \\ &\quad - nH \sum_{\alpha \neq n+1} h_{im}^\alpha h_{mj}^{n+1} h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{ik}^{n+1} h_{km}^{n+1} h_{mj}^\alpha h_{ij}^\alpha. \end{aligned}$$

We put $S_{\alpha\beta} = \sum h_{ij}^\alpha h_{ij}^\beta$ for any $\alpha, \beta \neq n+1$. Then $(S_{\alpha\beta})$ is a $(p-1) \times (p-1)$ symmetric matrix. It can be assumed to be diagonal for a suitable choice of e_{n+2}, \dots, e_{n+p} . Set $S_\alpha = S_{\alpha\alpha}$. We then have $\tau^2 = \sum S_\alpha$. In general, for a matrix $A = (a_{ij})$, we define $N(A) = \text{tr}(A^t A)$. Hence we get

$$\begin{aligned} &\sum_{\alpha, \beta \neq n+1} h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta h_{ij}^\alpha - 2 \sum_{\alpha, \beta \neq n+1} h_{ik}^\beta h_{km}^\alpha h_{mj}^\beta h_{ij}^\alpha \\ &\quad + \sum_{\alpha, \beta \neq n+1} h_{im}^\alpha h_{mk}^\beta h_{kj}^\beta h_{ij}^\alpha + \sum_{\alpha, \beta \neq n+1} h_{jm}^\alpha h_{mk}^\beta h_{ki}^\beta h_{ij}^\alpha \\ &= \sum_{\alpha \neq n+1} (S_\alpha)^2 + \sum_{\alpha, \beta \neq n+1} N(H^\alpha H^\beta - H^\beta H^\alpha). \end{aligned}$$

Obviously, we see

$$(5.2) \quad \sum_{\alpha, \beta \neq n+1} N(H^\alpha H^\beta - H^\beta H^\alpha) \geq 0.$$

Suppose $p \geq 2$. Let

$$(p-1)\sigma_1 = \tau^2 = \sum S_\alpha,$$

$$(p-1)(p-2)\sigma_2 = 2 \sum_{\alpha < \beta, \alpha, \beta \neq n+1} S_\alpha S_\beta.$$

Then we get

$$\sum (S_\alpha)^2 = (p-1)(\sigma_1)^2 + (p-1)(p-2)\{(\sigma_1)^2 - \sigma_2\},$$

$$(p-1)^2(p-2)\{(\sigma_1)^2 - \sigma_2\} = \sum_{\alpha < \beta, \alpha, \beta \neq n+1} (S_\alpha - S_\beta)^2.$$

Hence we obtain

$$(5.3) \quad \sum_{\alpha, \beta \neq n+1} h_{km}^\alpha h_{mk}^\beta h_{ij}^\beta h_{ij}^\alpha - 2 \sum_{\alpha, \beta \neq n+1} h_{ik}^\beta h_{km}^\alpha h_{mj}^\beta h_{ij}^\alpha$$

$$+ \sum_{\alpha, \beta \neq n+1} h_{im}^\alpha h_{mk}^\beta h_{kj}^\beta h_{ij}^\alpha + \sum_{\alpha, \beta \neq n+1} h_{ik}^\beta h_{km}^\alpha h_{mj}^\beta h_{ij}^\alpha$$

$$\geq (p-1)(\sigma_1)^2 = \frac{1}{p-1} \tau^4.$$

Then the equations (5.1), (5.2) and (5.3) imply

$$\frac{1}{2} \Delta \tau^2 \geq n c \tau^2 + \frac{1}{p-1} \tau^4 + \sum_{\alpha \neq n+1} h_{km}^\alpha h_{mk}^{n+1} h_{ij}^{n+1} h_{ij}^\alpha - 2 \sum_{\alpha \neq n+1} h_{ik}^{n+1} h_{km}^\alpha h_{mj}^{n+1} h_{ij}^\alpha$$

$$+ \sum_{\alpha \neq n+1} h_{im}^\alpha h_{mk}^{n+1} h_{kj}^{n+1} h_{ij}^\alpha - n H \sum_{\alpha \neq n+1} h_{im}^\alpha h_{mj}^{n+1} h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{ik}^{n+1} h_{km}^{n+1} h_{mj}^\alpha h_{ij}^\alpha.$$

For a fixed index α , since $H^\alpha H^{n+1} = H^{n+1} H^\alpha$, we can choose $\{e_1, \dots, e_n\}$ such that

$$h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}, \quad h_{ij}^{n+1} = \lambda_i \delta_{ij}.$$

Then we get

$$\sum h_{km}^\alpha h_{mk}^{n+1} h_{ij}^{n+1} h_{ij}^\alpha - 2 \sum h_{ik}^{n+1} h_{km}^\alpha h_{mj}^{n+1} h_{ij}^\alpha + \sum h_{im}^\alpha h_{mk}^{n+1} h_{kj}^{n+1} h_{ij}^\alpha$$

$$- n H \sum h_{im}^\alpha h_{mj}^{n+1} h_{ij}^\alpha + \sum h_{ik}^{n+1} h_{km}^{n+1} h_{mj}^\alpha h_{ij}^\alpha$$

$$= (\sum \lambda_i \lambda_i^\alpha)^2 - n H \sum \lambda_i (\lambda_i^\alpha)^2.$$

We notice here that eigenvalues λ_i^α are bounded by (4.19). In order to estimate the last term on the above equation, the following property is prepared.

LEMMA 5.1. *Let a_1, \dots, a_n be real numbers satisfying $\sum a_i = 0$ and let b_1, \dots, b_n be also real numbers. Then we have*

$$\sum a_i (b_i)^2 \leq \sqrt{\frac{n-1}{n}} \sqrt{\sum (a_i)^2} \sum (b_i)^2,$$

where the equality holds if and only if the $n-1$ of a_i 's are equal with each other and the corresponding $n-1$ of b_i 's are equal to 0.

PROOF. We consider the function $f = \sum a_i (b_i)^2$ with constraint $\sum a_i = 0$, $\sum (a_i)^2 = a$ and $\sum (b_i)^2 = b$. Then there exists a critical point of f on \mathbb{R}^{2n} at which we have

$$(5.4) \quad (b_i)^2 + \mu_1 + 2\mu_2 a_i = 0, \quad 2a_i b_i + 2\mu_3 b_i = 0.$$

From (5.4) we get

$$\mu_1 = -\frac{1}{n}b,$$

and the critical value of f is equal to $-\mu_3 b = -2\mu_2 a$, and therefore we have

$$(5.5) \quad a_i = -\mu_3, \quad \text{or} \quad b_i = 0.$$

If $a_i = -\mu_3$ for any index i , then we get $f = 0$, because of $\sum a_i = 0$. If $a_i = -\mu_3$, $1 \leq i \leq m$ and $b_j = 0$, $m+1 \leq j \leq n$, then we have from (5.4)

$$2\mu_2 a_j = \frac{1}{n}b, \quad j = m+1, \dots, n.$$

If $\mu_2 = 0$, then $f = 0$. Without loss of generality, we may suppose $\mu_2 \neq 0$. Thus we see

$$a_j = \frac{b}{2n\mu_2}, \quad j = m+1, \dots, n,$$

which yields

$$(5.6) \quad m\mu_3 = (n-m)\frac{b}{2n\mu_2}.$$

From (5.4) and (5.5) it follows that we obtain

$$\mu_2 = \pm \frac{1}{2} \sqrt{\frac{n-m}{nm}} \frac{b}{\sqrt{a}},$$

which means that we obtain

$$|f| \leq \sqrt{\frac{n-1}{n}} \sqrt{\sum (a_i)^2} \sqrt{\sum (b_i)^2}.$$

If the equality holds, then we have $m=1$ and $a_j = \pm \sqrt{a/n(n-1)}$, $b_j = 0$, $2 \leq j \leq n$.

The converse is obvious. \square

According to Lemma 5.1 we have

$$\begin{aligned} & (\sum_i \lambda_i \lambda_i^\alpha)^2 - nH \sum_i \lambda_i (\lambda_i^\alpha)^2 \geq -nH \sum_i (\lambda_i - H) (\lambda_i^\alpha)^2 - nH^2 \sum_i (\lambda_i^\alpha)^2 \\ & = -nH (\sum_i \mu_i (\lambda_i^\alpha)^2 + H \sum_i (\lambda_i^\alpha)^2) \\ & \geq -nH \left(\sqrt{\frac{n-1}{n}} \sqrt{\sum (h_{ij}^{n+1})^2 - nH^2} + H \right) \text{tr}(H^\alpha)^2. \end{aligned}$$

The right hand side of the inequality above does not depend on the choice of frame fields. Therefore we have

$$\begin{aligned} & \sum_{\alpha \neq n+1} h_{km}^\alpha h_{mk}^{n+1} h_{ij}^{n+1} h_{ij}^\alpha - 2 \sum_{\alpha \neq n+1} h_{ik}^{n+1} h_{km}^\alpha h_{mj}^{n+1} h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{im}^\alpha h_{mk}^{n+1} h_{kj}^{n+1} h_{ij}^\alpha \\ & - nH \sum_{\alpha \neq n+1} h_{im}^\alpha h_{mj}^{n+1} h_{ij}^\alpha + \sum_{\alpha \neq n+1} h_{jm}^\alpha h_{mk}^{n+1} h_{ki}^{n+1} h_{ij}^\alpha \\ & \geq -nH \left(\sqrt{\frac{n-1}{n}} \sqrt{\sum (h_{ij}^{n+1})^2 - nH^2} + H \right) \tau^2. \end{aligned}$$

Thus we have

$$\frac{1}{2} \Delta \tau^2 \geq \left\{ nc - nH \left(\sqrt{\frac{n-1}{n}} \sqrt{\sum (h_{ij}^{n+1})^2 - nH^2} + H \right) \right\} \tau^2 + \frac{1}{p-1} \tau^4.$$

Making use of the same proof as in the proof of $|\mu|^2$ above, we have

$$0 \geq \left\{ nc - nH \left(\sqrt{\frac{n-1}{n}} \sqrt{\sum (h_{ij}^{n+1})^2 - nH^2} + H \right) \right\} \tau^2 + \frac{1}{p-1} \tau^4.$$

Thus from (4.19) we get

$$(5.7) \quad \sup \tau^2 \leq (p-1) \left\{ nH \left(\sqrt{\frac{n-1}{n}} \sqrt{S_+(1) - nH^2} + H \right) - nc \right\}.$$

The equality (4.6), the inequalities (4.19) and (5.7) yield

$$S \leq S_+(p) + (p-1)H \{ nH + \sqrt{n(n-1) \{ S_+(1) - nH^2 \}} \}.$$

Hence we complete the proof of Theorem 1. \square

REMARK. When M is maximal (i. e., $H=0$), Theorem 1 implies $S \leq -n\mu c$. Ishihara [8] obtained this relation for complete maximal space-like submanifolds. When $p=1$, Theorem 1 becomes $S \leq S_+(1)$. This result is obtained by the first author and Nakagawa [6]. Hence Theorem 1 generalizes the results above.

6. Proof of Theorems 2 and 3.

Let M be an n -dimensional complete space-like submanifold with parallel mean curvature vector of $M_p^{n+p}(c)$, $c \leq 0$. We assume $S = S_+(p) + K(p)$. Then the equalities of all inequalities in the previous sections have to hold. Consequently, from (4.8) and (5.7) it is seen that

$$(6.1) \quad h_{ik}^\alpha = 0$$

for any i, j, k and α . Also from (4.2) and (5.7) it follows that

$$H^\alpha H^\beta = H^\beta H^\alpha$$

for any α and β . The equations imply that all of H^α are simultaneously

diagonalizable and the normal connection in the normal bundle of M is flat. Hence we can choose a suitable basis $\{e_i\}$ such that

$$(6.2) \quad h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$$

for any i, j and α . The submanifold M is said to be *isoparametric* [13] if the normal connection is flat and the characteristic polynomial of the shape operator A_ξ has constant coefficients over the domain of any local parallel normal field ξ .

LEMMA 6.1. *M is isoparametric.*

PROOF. Since the normal connection is flat, it is seen that there exist locally p mutually orthogonal unit normal vector fields which are parallel in the normal bundle. So we can choose a suitable parallel basis $\{e_\alpha\}$ and then we have $\omega_{\alpha\beta}=0$. Hence, since we have

$$(6.3) \quad \sum h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum h_{kj}^\alpha \omega_{ki} - \sum h_{ik}^\alpha \omega_{kj} + \sum h_{ij}^\beta \omega_{\beta\alpha},$$

setting $i=j$ in the above equation and using (6.1) we get $dh_{ii}^\alpha=0$. Hence h_{ii}^α is constant and M is isoparametric. \square

LEMMA 6.2. *M is of non-positive curvature.*

PROOF. Suppose that there exist indices i, j and α such that $h_{ii}^\alpha \neq h_{jj}^\alpha$. From the equation (6.3) we get

$$\sum h_{kji}^\alpha \omega_k + \sum h_{ik}^\alpha \omega_{kj} = (h_{ii}^\alpha - h_{jj}^\alpha) \omega_{ij} = 0,$$

from which it follows that $\omega_{ij}=0$. Accordingly, we have

$$\sum \omega_{ik} \wedge \omega_{kj} = 0.$$

In fact, for any fixed indices i and α we denote by $[i]$ the set consisting of indices k such that $h_{ii}^\alpha = h_{kk}^\alpha$. Then we have $[i] \neq [j]$ by the supposition and hence we get

$$\sum_k \omega_{ik} \wedge \omega_{kj} = \sum_{k \in [i]} \omega_{ik} \wedge \omega_{kj} + \sum_{k \in [j]} \omega_{ik} \wedge \omega_{kj} + \sum_{k \notin [i] \cup [j]} \omega_{ik} \wedge \omega_{kj},$$

each term of which vanishes identically. By the structure equation

$$d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l,$$

we obtain

$$R_{ijji} = c - \sum_{\beta} \lambda_i^\beta \lambda_j^\beta = 0.$$

Next, suppose that $h_{ii}^\alpha = h_{jj}^\alpha$ for any distinct indices i and j and for any index α . Then the Gauss equation implies

$$R_{iiji} = c - \sum_{\alpha} (h_{ii}^\alpha)^2 = c - \sum_{\alpha} (\lambda_i^\alpha)^2 \leq 0,$$

because of $c \leq 0$.

Thus M is of non-positive curvature. \square

PROOF OF THEOREM 2. By a theorem due to Koike [10] and Lemmas 6.1 and 6.2 it is seen that M is locally congruent to the product submanifold $H^{n_1}(c_1) \times \cdots \times H^{n_q}(c_q) \times R^m$ of R_q^{n+q} , where $\sum_{r=1}^q n_r + m = n$ and $1 \leq q \leq p$. Then M can be naturally regarded as the space-like submanifold of R_p^{n+p} whose mean curvature vector is given by (2.1). It is also parallel in the normal bundle of M in R_p^{n+p} . The constant $S_+(1)$ and the squared norm S of the second fundamental form are given by (2.2). Therefore it is seen that we have

$$S_+(p) + K(p) = -p \sum_{r=1}^q n_r^2 c_r = S,$$

which implies $p=q=1$ and $n_1=1$. Accordingly the hyperbolic cylinder $H^1(c_1) \times R^{n-1}$ of R_1^{n+1} is the complete connected space-like hypersurface with constant mean curvature whose squared norm S attaining the maximal value. \square

PROOF OF THEOREM 3. When $p=1$ it is seen by a theorem due to Ki, Kim and Nakagawa [9] that the hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$ is the complete spacelike hypersurface with constant mean curvature of $H_1^{n+1}(c)$ satisfying the given condition.

Suppose next that $p \geq 2$. By means of Koike's theorem and Lemmas 6.1 and 6.2 again, M is locally congruent to the product submanifold $H^{n_1}(c_1) \times \cdots \times H^{n_{q+1}}(c_{q+1})$ in $H_q^{n+q}(c')$, where $\sum_{r=1}^{q+1} n_r = n$, $\sum_{r=1}^{q+1} (1/c_r) = (1/c') \geq (1/c)$ and $H_q^{n+q}(c')$ is a totally umbilic submanifold of $H_p^{n+p}(c)$. The mean curvature vector of M in $H_q^{n+q}(c')$ is denoted by h' , which is parallel in the normal bundle of M in $H_q^{n+q}(c')$. Then the mean curvature vector h of M of $H_p^{n+p}(c)$ is given by $h = h' + h''$, where h'' is the mean curvature vector of $H_q^{n+q}(c')$ in $H_p^{n+p}(c)$. Consequently the mean curvature vector h is parallel in the normal bundle $N(M)$ and the mean curvature H and the squared norm S of M in $H_p^{n+p}(c)$ are given by

$$h^2 = n^2 H^2 = n^2 c - \sum_{r=1}^{q+1} n_r^2 c_r,$$

$$S = nc - \sum_{r=1}^{q+1} n_r c_r.$$

We have $S_+(1) \geq h^2 - nc$, because of $c < 0$. So it is seen by Lemma 2.1 that we obtain

$$(6.4) \quad S_+(p) + K(p) - S \geq h^2 - pnc + (p-1)h^2 - S = ph^2 - pnc - S \geq 0,$$

where the equality holds if and only if $H=0$. Accordingly, if we have $S = S_+(p) + K(p)$, then H must vanish identically. This implies that Theorem 3 is proved by a theorem due to Ishihara [8]. \square

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Institute of Mathematics, Fudan University, Shanghai 200433, P. R. China

Institute of Mathematics, University of Tsukuba, 305 Ibaraki, Japan